

SLOWLY ROTATING KERR BLACK HOLE AS A SOLUTION OF EINSTEIN-CARTAN GRAVITY EXTENDED BY A CHERN-SIMONS TERM

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We consider the nondynamical Chern-Simons (CS) modification to General Relativity (GR) in the framework of the Einstein-Cartan formulation, as providing a way to incorporate a slowly rotating Kerr black hole in the space of solutions. Our proposal lies on considering the CS term as a source of torsion and on an iterative procedure to look for vacuum solutions of the system, by expanding the tetrad, the connection and the embedding parameter, in powers of a dimensionless small parameter β which codifies the CS coupling. Starting from a torsionless zeroth-order vacuum solution we derive the second-order differential equation for the $\mathcal{O}(\beta)$ corrections to the metric, for an arbitrary embedding parameter. Furthermore we can show that the slowly rotating Kerr metric is an $\mathcal{O}(\beta)$ solution of the system either in the canonical or the axial embeddings.

1. Introduction

The CS modification of GR was introduced in the framework of the Einstein-Hilbert formulation of GR in Ref. 1. The model, referred to as CS-EH, has been the subject of numerous investigations in the literature² and further understood as providing a way to describe gravitational parity violation. It can also be viewed as the GR analogue of the Carroll-Field-Jackiw version of extended electrodynamics by a Lorentz-symmetry breaking term of the Chern-Simons kind,³ which is one of the many terms of

the mSME. On the other hand it could be a means to implement a general relativistic version of Cohen and Glashow's Very Special Relativity⁴ after it was pointed out that the symmetries of CS extended electrodynamics are those of one of the VSR models.⁵ Despite the many successes of this model and the fact that some slowly rotating black hole solutions have been found,⁶⁻⁹ there remains an important unsolved question as to whether or not the Kerr black hole is a solution of a CS modified gravity.

To this end we adhere to the viewpoint of treating the CS extension of GR in the Einstein-Cartan (EC) formulation as done by Botta Cantcheff,¹⁰ which we will refer to as CS-EC. Namely, the CS coupling is implemented in terms of the SO(1,3) gauge connection of GR, providing a source of torsion even in the absence of matter,^{11,12} thus putting torsion at the 'forefront.'¹³

The action reads:

$$S[e^c, \omega^a_b] = \frac{\kappa}{2} \int (\mathbf{R}_{ab} \wedge^* (\mathbf{e}^a \wedge \mathbf{e}^b) + \beta \vartheta \mathbf{R}^a_b \wedge \mathbf{R}^b_a), \quad (1)$$

where the Riemann \mathbf{R}_{ab} and torsion \mathbf{T}^a two-forms are defined as:

$$\mathbf{R}^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b, \quad \mathbf{T}^a = d\mathbf{e}^a + \omega^a_b \wedge \mathbf{e}^b. \quad (2)$$

In Eq. (1) the variable ϑ , taken as nondynamical, is the so called embedding parameter and β is a dimensionless parameter, which serves as a bookkeeping device that will be clarified below.

The equations of motion for Eq. (1) are:

$$\begin{aligned} \delta \mathbf{e}^c : \epsilon_{abcd} \mathbf{R}^{ab}(\omega) \wedge \mathbf{e}^c &= 0, \\ \delta \omega^a_b : \epsilon_{abqp} \mathbf{e}^q \wedge \mathbf{T}^p + 2\beta d\vartheta \wedge \mathbf{R}_{ab}(\omega) &= 0. \end{aligned} \quad (3) \quad (4)$$

2. Perturbation scheme of the CS term as a torsion source

Equation (4) reveals the role of the CS term as a source of torsion and it can be solved for the tensor components of the torsion tensor in terms of the complete torsionful Riemann tensor to yield:

$$T^\sigma_{\alpha\beta} = \frac{\beta}{2 \det(e_\mu^a)} (\partial_\mu \vartheta) \left[2\epsilon^{\mu\nu\rho\sigma} R_{\alpha\beta\nu\rho} + (\delta^\sigma_\alpha \epsilon^{\mu\nu\rho\omega} R_{\beta\nu\rho\omega} - \delta^\sigma_\beta \epsilon^{\mu\nu\rho\omega} R_{\alpha\nu\rho\omega}) \right]. \quad (5)$$

Here $\epsilon^{\mu\nu\rho\omega} = 0, \pm 1$ is the Levi-Civita symbol. The LHS is in turn defined in terms of the spin connection's components which also enter the definition of the Riemann tensor; therefore the latter is a highly non-linear equation for the spin connection, which we do not aim to solve exactly for the moment.

Indeed, the severe observational constraints on the presence of torsion in Nature,¹⁴ which in our case is driven by the CS term, leads us to pose the following ansatz for the dynamical variables:

$$e_{\mu}^{c} = e_{\mu}^{(0)c} + \beta e_{\mu}^{(1)c} + \dots, \quad \omega_{\mu}^{a}{}^b = \omega_{\mu}^{(0)a}{}^b + \beta \omega_{\mu}^{(1)a}{}^b + \dots, \quad (6)$$

and similarly for the embedding parameter. All quantities such as the Riemann, torsion tensor, etc. . . are expanded accordingly and zeroth order is to be understood as torsion free. Say we start from a zeroth-order approximation which might be taken as any vacuum solution of Einstein equation with zero torsion, thus providing $e^{(0)}, \omega^{(0)}, T^{(0)} = 0$ and $R^{(0)}$. Then we obtain $T^{(1)}$ according to Eq. (5) and from $T^{(1)}$ we solve for $\omega^{(1)}$, arising from Eq. (2), in terms of $\partial e^{(1)}$ plus zeroth-order quantities including the zeroth-order covariant derivative. Next we construct the first-order Riemann tensor, according to Eq. (2). This will introduce an additional derivative to $\partial e^{(1)}$, such that the final equation $R_{\mu\nu}^{(1)} = 0$ will be of second order in the unknown $e^{(1)}$, providing the first correction to the metric $g_{\mu\nu}^{(1)}$, thus determining all first-order variables. Second- and higher-order corrections can be obtained iterating the procedure above.

3. Consistence between dynamics and geometry

Field equations and Bianchi identities together result in consistency conditions that must be met by the dynamical variables. For lack of space these can only be briefly discussed here. For example, the field equations together impose $\epsilon_a^{bcd} T_{bc}^a = 0$. Applying the exterior covariant derivative \mathcal{D} acting on Lorentz tensor valued p -forms¹⁵ on the field equations together with the Bianchi identities also written in terms of \mathcal{D} (e.g., $\mathcal{D} \wedge \mathbf{R}^a{}_b = 0$) demands the resulting (torsionful) Ricci tensor to be symmetric in its components, $R_{ab} = R_{ba}$. On the other hand, the corresponding Einstein equation in tensorial components reads $G^{\mu}{}_{\nu} = 0$, where the Einstein tensor is defined in terms of the torsionful Riemann tensor and therefore $R^{\mu}{}_{\nu} = 0$. Finally from the first Bianchi identity commented above, we can form $\epsilon_{abqp} \mathcal{D} \wedge (\mathbf{R}^{ab}(\omega)) \wedge \mathbf{e}^c = 0$, leading to:

$$\nabla_{\alpha} G^{\alpha}{}_{\psi} - T^{\theta}{}_{\psi\beta} R^{\beta}{}_{\theta} - \frac{1}{2} T^{\beta}{}_{\alpha\theta} R^{\alpha\theta}{}_{\beta\psi} = 0, \quad (7)$$

which due to the vanishing of $R^{\mu}{}_{\nu}$ implies:

$$T^{\beta}{}_{\alpha\theta} R^{\alpha\theta}{}_{\beta\psi} = 0. \quad (8)$$

We can only mention that, for the cases discussed below, all the necessary consistency conditions of the kind described above are indeed satisfied at least to the desired $\mathcal{O}(\beta)$ level.

4. Slowly rotating Kerr solution

Solving the field equations to first order in β as outlined above leads to:

$$\frac{\vartheta^{(0)}_{;\lambda;\alpha}}{e^{(0)}} \left(\epsilon^{\lambda\delta\gamma}{}_{\mu} R^{(0)}{}^{\alpha}{}_{\delta\gamma\nu} + \epsilon^{\lambda\delta\gamma}{}_{\nu} R^{(0)}{}^{\alpha}{}_{\delta\gamma\mu} \right) = g^{(1)}_{\nu\alpha;\mu} + g^{(1)}_{\mu\alpha;\nu} - g^{(1)}_{\mu\nu;\alpha} - g^{(1)}_{;\mu;\nu}, \quad (9)$$

where $g^{(1)}_{\mu\nu} \equiv e^{(1)}_{\mu\nu} + e^{(1)}_{\nu\mu}$, $g^{(1)} \equiv g^{(0)\mu\nu} g^{(1)}_{\mu\nu} = 2e^{(1)\rho}{}_{\rho}$, $e^{(0)} \equiv \det(e^{(0)}_{\mu}{}^a)$ and the semicolon denotes zeroth-order covariant derivative. It is verified that for the canonical embedding $\vartheta^{(0)} = t/\mu$ and for $g^{(0)}_{\mu\nu}$ given by the Schwarzschild geometry, the LHS of Eq. (9) vanishes and $g^{(1)}_{\mu\nu} \equiv 0$ solves the remaining RHS. Also, the consistency conditions of Sec. 3 are satisfied too at least to $\mathcal{O}(\beta)$. Therefore the Schwarzschild geometry is an $\mathcal{O}(\beta)$ solution of the theory. However, though it is not a trivial solution, it can also be verified that $g^{(1)}_{\mu\nu} = -(2M^2/r) \sin^2 \theta \delta^t_{\mu} \delta^{\phi}_{\nu}$ solves the equation and satisfies the consistency conditions too. Thus we have found an $\mathcal{O}(\beta)$ consistent solution which to first order reads $g_{\mu\nu} = g^{(0)}_{\mu\nu} + \beta g^{(1)}_{\mu\nu}$. Identifying $\beta \equiv a/M$, where M is an spherical body's mass and a its angular momentum we can interpret the above solution as describing a slowly rotating Kerr black hole if $a/M \ll 1$,

$$g_{\mu\nu} \approx g^{\text{Schw}}_{\mu\nu} - \frac{2Ma}{r} \sin^2 \theta \delta^t_{\mu} \delta^{\phi}_{\nu} = g^{\text{slow Kerr}}_{\mu\nu}. \quad (10)$$

5. Final comments

In the language of the SME, the CS term ϑ can be viewed as an externally prescribed explicit symmetry breaking quantity. Although this may lead to inconsistencies pointed out in Ref. 16, here our first concern is to explore the consequences of taking torsion into account in CS extended gravity. The case of a dynamical ϑ that undergoes spontaneous symmetry breaking is clearly more interesting but will be dealt with elsewhere. One of the important consequences that our investigation shows is the possibility to accommodate the Kerr solution into the theory. This may be considered as one of the flaws of the theory in its original formulation, or at least a missing ingredient. Thus our result can be considered as an important contribution

in this field, pointing towards the benefits of not limiting attention to spaces of solution with a symmetric connection.

It is interesting to note that the above solution is still valid for $\vartheta^{(0)} = Mr \cos \theta$, which we call an ‘axial’ embedding, producing a symmetry breaking direction parallel to the rotation axis of the Kerr metric, i.e., suggesting that the slowly rotating Kerr black hole could be interpreted as arising from the breaking of spherical to axial symmetry.

It is also true that the perturbative expansion here presented may be valid only up to a given order, which needs to be understood further. Nevertheless, this problem is not new in this context, for example, in a previous formulation of CS-EC even the Schwarzschild solution was valid up to a given order only.¹⁰

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References

1. R. Jackiw and S.Y. Pi, Phys. Rev. D **68**, 104012 (2003).
2. S. Alexander and N. Yunes, Phys. Rep. **480**, 1 (2009).
3. S.M. Carroll, G.B. Field and R. Jackiw, Phys. Rev. D **41**, 1231 (1990).
4. A.G. Cohen and S.L. Glashow, Phys. Rev. Lett. **97**, 021601 (2006).
5. A.J. Hariton and R. Lehnert, Phys. Lett. A **367**, 11 (2007).
6. K. Konno, T. Matsuyama and S. Tanda, Phys. Rev. D **76**, 024009 (2007).
7. D. Grumiller and N. Yunes, Phys. Rev. D **77**, 044015 (2008).
8. K. Konno, T. Matsuyama and S. Tanda, Prog. Theor. Phys. **122**, 561 (2009).
9. N. Yunes and F. Pretorius, Phys. Rev. D **79**, 084043 (2009).
10. M. Botta Cantcheff, Phys. Rev. D **78**, 025002 (2008); PoS **IC2006**, 060 (2006).
11. F.W. Hehl, J.D. McCrea, E.W. Mielke and Y. Ne’eman, Phys. Rep. **258**, 1 (1995).
12. I.L. Shapiro, Phys. Rep. **357**, 113 (2002).
13. See J. Bjorken’s and I.L. Shapiro’s contributions to these Proceedings.
14. V.A. Kostelecký, N. Russell and J. Tasson, Phys. Rev. Lett. **100**, 111102 (2008).
15. M. Goeckeler and T. Schuecker, *Differential Geometry, Gauge Theories, and Gravity*, Cambridge University Press, 1987.
16. V.A. Kostelecký, Phys. Rev. D **69**, 105009 (2004).